Classification of Facets of the Hop Constrained Chain Polytope using Projection

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Paths/Walks/Chains

Given

- digraph $D = (N, A)$: obtained from the complete digraph on $N = \{s, t, 1, 2, \ldots, n\}$ by deleting the arcs $a \in \delta^-(s) \cup \delta^+(t) \cup \{(s,t)\}$.
  Here, for any node $v \in N$, $\delta^+(v)$ and $\delta^-(v)$ denote the set of arcs leaving and entering v, respectively.
- length function $d : A \to \mathbb{R}$.
- $k \in \{1, \ldots, n\}$.

▷ (s,t)-chain: node-arc-sequence starting in s and ending in t.

▷ (s,t)-walk: node-arc-sequence starting in s and ending in t such that each arc is used at most one time.

▷ (s,t)-path: node-arc-sequence starting in s and ending in t such that each node of D is visited at most one time.

Hop constrained paths/walks/chains

Hop constrained shortest path problem: (HCSPP)
Find an (s, t)-path with at most k arcs of minimum length.
Complexity: NP-hard

Hop constrained shortest walk problem: (HCSWP)
Find an (s, t)-walk with at most k arcs of minimum length.
Complexity: ?

Hop constrained shortest chain problem: (HCSCP)
Find an (s, t)-chain with at most k arcs of minimum length.
Complexity: polynomial (Moore-Bellman-Ford algorithm)

6 polyhedra

Hop constrained path polytope:
$P^{\leq k}_{s,t\text{-path}} := \text{conv}\{\chi^P \in \mathbb{R}^A : P\ (s,t)\text{-path},\ |P| \leq k\}$.
Its dominant: $\text{dmt}(P^{\leq k}_{s,t\text{-path}}) := P^{\leq k}_{s,t\text{-path}} + \mathbb{R}^A_+$

Hop constrained walk polytope:
$P^{\leq k}_{s,t\text{-walk}} := \text{conv}\{\chi^W \in \mathbb{R}^A : P\ (s,t)\text{-walk},\ |W| \leq k\}$.
Its dominant: $\text{dmt}(P^{\leq k}_{s,t\text{-walk}}) := P^{\leq k}_{s,t\text{-walk}} + \mathbb{R}^A_+$

Hop constrained chain polytope:
$P^{\leq k}_{s,t\text{-chain}} := \text{conv}\{\chi^C \in \mathbb{R}^A : P\ (s,t)\text{-chain},\ |C| \leq k\}$.
Its dominant: $\text{dmt}(P^{\leq k}_{s,t\text{-chain}}) := P^{\leq k}_{s,t\text{-chain}} + \mathbb{R}^A_+$

6 polyhedra and their relations to each other

Lemma 1

(i) $P^{\leq k}_{s,t\text{-}path} \subseteq P^{\leq k}_{s,t\text{-}walk} \subseteq P^{\leq k}_{s,t\text{-}chain}$.

(ii) $dmt(P^{\leq k}_{s,t\text{-}path}) = dmt(P^{\leq k}_{s,t\text{-}walk}) = dmt(P^{\leq k}_{s,t\text{-}chain})$.

(iii) $P^{\leq k}_{s,t\text{-}chain} \subseteq dmt(P^{\leq k}_{s,t\text{-}chain})$.

Characterization of the integer points of $P^{\leq k}_{s,t\text{-path}}$

$$x(\delta^+(i)) - x(\delta^-(i)) = \begin{cases} 1 \text{ if } i = s, \\ 0 \text{ if } i \in N \setminus \{s,t\}, \\ -1 \text{ if } i = t, \end{cases} \quad (1)$$

$$x(A) \leq k, \quad (2)$$

$$x(\delta^+(i)) \leq 1 \qquad \forall\, i \in N \setminus \{s,t\}, \quad (3)$$

$$x(\delta^+(S)) \geq x(\delta^+(j)) \qquad \forall\, S \subset N, \; 3 \leq |S| \leq n-2, \; s,t \in S, j \in N \setminus S, \quad (4)$$

$$x_{ij} \in \{0,1\} \qquad \forall\, (i,j) \in A. \quad (5)$$

Some facts

Complete linear descriptions are known for

- the dominant of the ordinary path polytope $\mathrm{dmt}(P_{s,t\text{-path}})$, where $P_{s,t\text{-path}} := P^{\leq n+1}_{0s,t\text{-path}}$ (see, for instance, Schrijver (2003));
- the 3-hop constrained path polytope $P^{\leq 3}_{s,t\text{-path}}$ (Dahl and Gouveia (2004));
- the 4-hop constrained walk polytope $P^{\leq 4}_{s,t\text{-walk}}$ (Dahl, Foldnes, and Gouveia (2004)).

Can be given by inequalities with coefficients in $\{-1, 0, 1\}$.

r-jump inequalities

Theorem 2 (Dahl, Foldnes, and Gouveia (2004))

Let

$$N = \bigcup_{p=0}^{k+r} S_p$$

be a partition of node set N, *where* $r \in \mathbb{N}$, $1 \leq r \leq n - k$, $S_0 = \{s\}$, *and* $S_{k+r} = \{t\}$. *For* $k \geq 4$, *the* r-jump inequality

$$\sum_{p=0}^{k+r-1} \sum_{q=p+1}^{k+r} \alpha_{pq} x((S_p : S_q)) \geq r, \tag{6}$$

where for $p < q$, $\alpha_{pq} := \min\{q - p - 1, r\}$, *is facet defining for* $dmt(P^{\leq k}_{s,t\text{-}path})$.

r-jump inequalities

Illustration for $r = 1$ and $k = 5$.

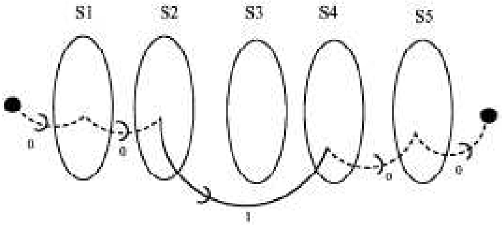

Figure: Picture from Dahl, Foldnes, and Gouveia

r-jump inequalities

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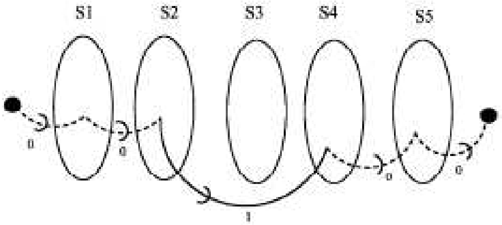

Figure: Picture from Dahl, Foldnes, and Gouveia

Inequalities (6) define facets of $\text{dmt}(P^{\leq k}_{s,t\text{-chain}})$ but not of $P^{\leq k}_{s,t\text{-path}}$, $P^{\leq k}_{s,t\text{-walk}}$, and $P^{\leq k}_{s,t\text{-chain}}$.

Our contribution

We present

- a classification of all 0/1-facet defining inequalities for $\text{dmt}(P^{\leq k}_{s,t\text{-chain}})$.
- a classification of all -1/0/1-facet defining inequalities for $P^{\leq k}_{s,t\text{-chain}}$.
- a generalization of r-jump inequalities.
- a systematic way to lift r-jump inequalities or their generalization into facet defining inequalities for $P^{\leq k}_{s,t\text{-chain}}$.

Algorithm 1: Moore-Bellman-Ford

Input: A digraph $D = (N, A)$, a fixed node $s \in N$, and a length function $d : A \rightarrow \mathbb{R}$.

Output: For each node $j \in N$ and each number $\ell \in \{0, \ldots, |N|-1\}$, the length $u_j^{(\ell)}$ of a shortest (s, j)-chain using at most ℓ arcs and its predecessor $p(j, \ell)$ on such a chain. If j is not reachable from s, then $u_j^{(\ell)} = +\infty$ and $p(j, \ell)$ is undefined for all ℓ.

(1) Set $u_s^{(0)} := 0$ and $u_j^{(0)} := +\infty$ for all $j \in N \setminus \{s\}$.
(2) **for** $\ell := 1$ **to** $|N| - 1$ **do**
  Set $t_j := u_j^{(\ell-1)}$ for all $j \in N$.
  forall $(i, j) \in A$ **do**
    if $t_j > u_i^{(\ell-1)} + d((i, j))$ **then**
      Set $t_j := u_i^{(\ell-1)} + d((i, j))$ and $p(j, \ell) := i$.
  Set $u_j^{(\ell)} := t_j$ for all $j \in N$.

Costruction of the DP-graph

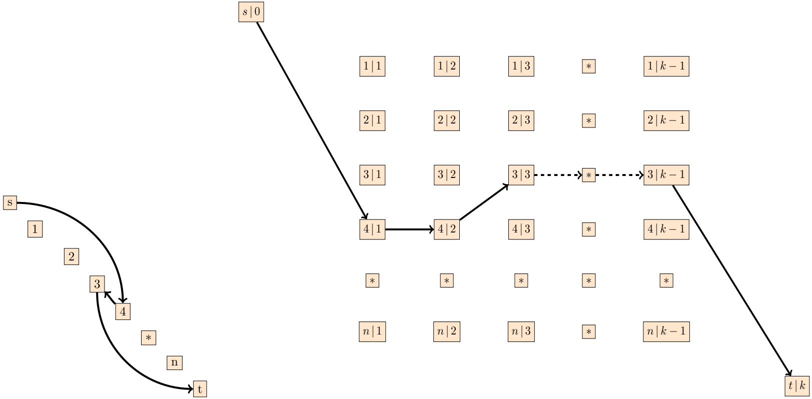

Figure: Digraph D and DP-graph $\mathcal{D} = (\mathcal{N}, \mathcal{A})$ associated with (D, s, t, k); arc sets are omitted. Illustration of an (s, t)-k-walk and one of its counterparts in $\mathcal{D}$.

Projection I

Given $Q := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : Ay + Bx \geq b\}$,

- the *projection of Q onto the x-space* is defined as

$$\text{Proj}_x(Q) := \{x \in \mathbb{R}^p : \exists y \in \mathbb{R}^q \text{ with } (x, y) \in Q\}.$$

Extended formulations and projection

Theorem 3

$P^{\leq k}_{s,t\text{-chain}}$ ($dmt(P^{\leq k}_{s,t\text{-chain}})$) is the projection of the polytope constituted by

$$y(\delta^+([s,0])) = 1, \qquad (7)$$

$$y(\delta^-([t,k])) = 1, \qquad (8)$$

$$y(\delta^-([i,\ell])) - y(\delta^+([i,\ell])) = 0 \quad \text{for all } [i,\ell] \in \mathcal{N}', \qquad (9)$$

$$y_a \geq 0 \quad \text{for all } a \in \mathcal{A}, \qquad (10)$$

$$x = (\geq) Ty \qquad (11)$$

onto the x-space, where $\mathcal{N}' := \mathcal{N} \setminus \{[s,0],[t,k]\}$ and T represents the set function

$$\varphi : \mathcal{A} \to A \cup \varnothing,\ \varphi(([i,h],[j,\ell])) = \begin{cases} (i,j) & \text{if } i \neq j \\ \varnothing & \text{else.} \end{cases}$$

Projection II

Given $Q := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : Ay + Bx \geq b\}$,

- the *projection of Q onto the x-space* is defined as

$$\text{Proj}_x(Q) := \{x \in \mathbb{R}^p : \exists y \in \mathbb{R}^q \text{ with } (x, y) \in Q\}.$$

- $\mathcal{C} := \{v : v^T A = 0^T, v \geq 0\}$ is called the *projection cone*.

Theorem 4 (Benders)

$$Proj_x(Q) = \{x \in \mathbb{R}^p : (v^T B)x \geq v^T b, v \in extr(\mathcal{C})\},$$

where $extr(\mathcal{C})$ *denotes the set of extreme rays of* $\mathcal{C}$*.*

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Usually, the goal is to determine all or at least a subset of all extreme rays $v \in \text{extr}(\mathcal{C})$ whose corresponding inequalities $(v^T B)x \geq v^T b$ define facets of $\text{Proj}_x(Q)$.

The projection cones

The projection cone $\mathcal{C}^{=}$ associated with $P^{\leq k}_{s,t\text{-chain}}$ $(\mathcal{C}^{\geq}, \text{dmt}(P^{\leq k}_{s,t\text{-chain}}))$ is the set of all $(\pi, \rho, \sigma) \in \mathbb{R}^{\mathcal{N}} \times \mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{A}$ satisfying the equations

$$\begin{aligned} \pi_{i,\ell-1} - \pi_{i\ell} - \rho_a &= 0 \qquad \forall\, a = ([i,\ell-1],[i,\ell]) \in \hat{\mathcal{A}}, \\ \sigma_{ij} + \pi_{i,\ell-1} - \pi_{j\ell} - \rho_a &= 0 \qquad \forall\, a = ([i,\ell-1],[j,\ell]) \in \mathcal{A} \setminus \hat{\mathcal{A}}, \\ \rho \geq 0 \quad (\sigma &\geq 0) \end{aligned}$$

Here,
$\hat{\mathcal{A}} := \{a = ([i,\ell-1],[i,\ell]) : i \in N \setminus \{s,t\}, \ell \in \{2,\ldots,k-1\}\}$.
Moreover, we define $\mathcal{A}^{ij} := \{a \in \mathcal{A} : \varphi(a) = (i,j)\}$.

Valid inequality: $\sum_{a \in A} \sigma_a x_a \geq \pi_{tk} - \pi_{s0}$.

The projection cones

The variables ρ_a, $a \in \mathcal{A}$, only act as slack variables. Projecting them out, we see that every $(\pi, \rho, \sigma) \in \mathcal{C}^{=}$ $(\mathcal{C}^{\geq})$ satisfies the inequalities

$$\pi_{i,\ell-1} - \pi_{i\ell} \geq 0 \qquad \forall\, i \in N \setminus \{s,t\},\ \ell = 2,\dots,k-1 \quad (12)$$

$$\sigma_{ij} + \pi_{i,\ell-1} - \pi_{j\ell} \geq 0 \qquad \forall\, a = ([i,\ell-1],[j,\ell]) \in \mathcal{A} \setminus \hat{\mathcal{A}} \quad (13)$$

$$(\sigma \geq 0). \quad (14)$$

The projection cones

For fixed π satisfying (12), denote by $\mathcal{C}_\pi^=$ ($\mathcal{C}_\geq^=$) the set of all $\sigma \in \mathbb{R}^A$ satisfying (13) (and (14)). Let $\tau := \pi_{tk} - \pi_{s0}$.

- $\sigma \in \mathcal{C}_\pi^= (\in \mathcal{C}_\pi^\geq) \Rightarrow$ right hand side $= \tau$.
- $\sigma \in \mathcal{C}_\pi^= (\in \mathcal{C}_\pi^\geq) \Rightarrow$

$$\sigma_{ij} \geq \max(\pi_{j\ell} - \pi_{i,\ell-1}) \qquad (\text{and } \sigma_{ij} \geq 0) \quad \text{for all } (i,j) \in A.$$

The projection cones

Conversely, define σ^π by

$$\sigma^\pi_{ij} := \max\{\pi_{j\ell} - \pi_{i,\ell-1} : a = ([i,\ell-1],[j,\ell]) \in \mathcal{A}^{ij}\}$$

for all $(i,j) \in A$. Then, $\sigma^\pi \in \mathcal{C}^=_\pi$ and $\sigma^\pi \leq \sigma$ for all $\sigma \in \mathcal{C}^=_\pi$. Thus, σ^π provides the strongest valid inequality under all inequalities $\sigma^T x \geq \pi_{tk} - \pi_{s0}$ with $\sigma \in \mathcal{C}^=_\pi$.

The projection cones

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Analogous for $\sigma^{\pi,+} \in \mathcal{C}^\geq_\pi$ defined by

$$\sigma^{\pi,+}_{ij} := \max\{0, \pi_{j\ell} - \pi_{i,\ell-1} : a = ([i,\ell-1],[j,\ell]) \in \mathcal{A}^{ij}\}$$

for $(i,j) \in A$.

The projection cones

- $F(\pi)$: face of $P^{\leq k}_{s,t\text{-chain}}$ induced by $\sum_{a \in A} \sigma^{\pi}_a x_a \geq \pi_{tk} - \pi_{s0}$.
- $F(\pi, +)$: face of $\text{dmt}(P^{\leq k}_{s,t\text{-chain}})$ induced by $\sum_{a \in A} \sigma^{\pi,+}_a x_a \geq \pi_{tk} - \pi_{s0}$.

Example

$$\pi = \begin{pmatrix} 0 & & & & \\ & 2 & 1 & 1 & \\ & 4 & 1 & 0 & \\ & 2 & 2 & 2 & \\ & 1 & 1 & 0 & \\ & & & & 3 \end{pmatrix}$$

$F(\pi)$:

$F(\pi, +)$:

Example

$$\pi = \begin{pmatrix} 0 & & & & \\ & 2 & 1 & 1 & \\ & 4 & 1 & 0 & \\ & 2 & 2 & 2 & \\ & 1 & 1 & 0 & \\ & & & & 3 \end{pmatrix}$$

$F(\pi)$: $2x_{s1}$

$F(\pi, +)$: $2x_{s1}$

Example

$$\pi = \begin{pmatrix} 0 & & & & \\ & 2 & 1 & 1 & \\ & 4 & 1 & 0 & \\ & 2 & 2 & 2 & \\ & 1 & 1 & 0 & \\ & & & & 3 \end{pmatrix}$$

$F(\pi)$: $2x_{s1} + 4x_{s2}$

$F(\pi, +)$: $2x_{s1} + 4x_{s2}$

Example

$$\pi = \begin{pmatrix} 0 & & & & \\ & 2 & 1 & 1 & \\ & 4 & 1 & 0 & \\ & 2 & 2 & 2 & \\ & 1 & 1 & 0 & \\ & & & & 3 \end{pmatrix}$$

$F(\pi)$: $2x_{s1} + 4x_{s2} + 2x_{s3}$

$F(\pi, +)$: $2x_{s1} + 4x_{s2} + 2x_{s3}$

Example

$$\pi = \begin{pmatrix} \color{blue}{0} & & & & \\ & 2 & 1 & 1 & \\ & 4 & 1 & 0 & \\ & 2 & 2 & 2 & \\ & \color{red}{1} & 1 & 0 & \\ & & & & 3 \end{pmatrix}$$

$F(\pi): \quad 2x_{s1} + 4x_{s2} + 2x_{s3} + 1x_{s4}$

$F(\pi, +): \quad 2x_{s1} + 4x_{s2} + 2x_{s3} + 1x_{s4}$

Example

$$\pi = \begin{pmatrix} 0 & & & & \\ & 2 & 1 & 1 & \\ & 4 & 1 & 0 & \\ & 2 & 2 & 2 & \\ & 1 & 1 & 0 & \\ & & & & 3 \end{pmatrix}$$

$$F(\pi): \quad 2x_{s1} + 4x_{s2} + 2x_{s3} + 1x_{s4} - 1x_{12}$$

$$F(\pi, +): \quad 2x_{s1} + 4x_{s2} + 2x_{s3} + 1x_{s4} + 0x_{12}$$

Example

$$\pi = \begin{pmatrix} 0 & & & & \\ & \color{blue}{2} & \color{blue}{1} & 1 & \\ & 4 & 1 & 0 & \\ & 2 & \color{red}{2} & \color{red}{2} & \\ & 1 & 1 & 0 & \\ & & & & 3 \end{pmatrix}$$

$$F(\pi): \quad 2x_{s1} + 4x_{s2} + 2x_{s3} + 1x_{s4} - 1x_{12} + 1x_{13}$$

$$F(\pi, +): \quad 2x_{s1} + 4x_{s2} + 2x_{s3} + 1x_{s4} + 0x_{12} + 1x_{13}$$

Example

$$\pi = \begin{pmatrix} 0 & & & & \\ & 2 & 1 & 1 & \\ & 4 & 1 & 0 & \\ & 2 & 2 & 2 & \\ & 1 & 1 & 0 & \\ & & & & 3 \end{pmatrix}$$

$$F(\pi): \quad 2x_{s1} + 4x_{s2} + 2x_{s3} + 1x_{s4} - 1x_{12} + 1x_{13} - 1x_{14}$$

$$F(\pi,+): \quad 2x_{s1} + 4x_{s2} + 2x_{s3} + 1x_{s4} + 0x_{12} + 1x_{13} + 0x_{14}$$

Example

$$\pi = \begin{pmatrix} 0 & & & & \\ & 2 & 1 & 1 & \\ & 4 & 1 & 0 & \\ & 2 & 2 & 2 & \\ & 1 & 1 & 0 & \\ & & & & 3 \end{pmatrix}$$

$$F(\pi): \quad 2x_{s1} + 4x_{s2} + 2x_{s3} + 1x_{s4} - 1x_{12} + 1x_{13} - 1x_{14} + 0x_{21}$$

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Example

$$\pi = \begin{pmatrix} 0 & & & & \\ & 2 & 1 & 1 & \\ & 4 & 1 & 0 & \\ & 2 & 2 & 2 & \\ & 1 & 1 & 0 & \\ & & & & 3 \end{pmatrix}$$

$F(\pi)$: $2x_{s1} + 4x_{s2} + 2x_{s3} + 1x_{s4} - 1x_{12} + 1x_{13} - 1x_{14} + 0x_{21} + \ldots \geq 3$

$F(\pi, +)$: $2x_{s1} + 4x_{s2} + 2x_{s3} + 1x_{s4} + 0x_{12} + 1x_{13} + 0x_{14} + 0x_{21} + \ldots \geq 3$

Matrix manipulations

Definition 5

$\tilde{\pi}$ is called the *minimal representative of* π if $\tilde{\pi}$ is lexicographic minimal (w.r.t. row permutation), $\tilde{\pi}$ does not contain identical rows, and π can be obtained from $\tilde{\pi}$ by copying and/or permuting rows of $\tilde{\pi}$.

$$\pi = \begin{pmatrix} 0 & & & & \\ & 2 & 1 & 1 & \\ & 4 & 1 & 0 & \\ & 4 & 1 & 0 & \\ & 2 & 2 & 2 & \\ & 1 & 1 & 0 & \\ & 1 & 1 & 0 & \\ & & & & 3 \end{pmatrix}, \quad \tilde{\pi} = \begin{pmatrix} 0 & & & & \\ & 1 & 1 & 0 & \\ & 2 & 1 & 1 & \\ & 2 & 2 & 2 & \\ & 4 & 1 & 0 & \\ & & & & 3 \end{pmatrix}$$

Characterization of 0/1-facets of $\mathrm{dmt}(P^{\leq k}_{s,t\text{-chain}})$

Theorem 6

Let π be a 0/1-vector. Then, $F(\pi,+)$ is a facet of $\mathrm{dmt}(P^{\leq k}_{s,t\text{-chain}})$ if and only if the minimal representative of π is one of the following four vectors

$$\begin{pmatrix} 0 & & & & & \\ & 0 & \cdots & \cdots & 0 & \\ & 1 & \ddots & & \vdots & \\ & \vdots & \ddots & \ddots & \vdots & \\ & \vdots & & \ddots & 0 & \\ & 1 & \cdots & \cdots & 1 & \\ & & & & & 1 \end{pmatrix} \quad \Bigg| \quad \begin{pmatrix} 0 & & & & \\ & 0 & \dots & 0 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 0 & & & & \\ & 1 & \dots & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 0 & & & & \\ & 0 & \dots & 0 & \\ & 1 & \dots & 1 & \\ & & & & 1 \end{pmatrix}$$

$\Rightarrow$ *1-jump inequality* | $\Rightarrow$ *min-cut inequality*

Characterization of -1/0/1-facets of $P^{\leq k}_{s,t\text{-chain}}$

Theorem 7

Let $k \geq 4$, and let π be a 0/1-vector with $\pi_{i,k-1} = 0$ for $i = 1, 2, \ldots, n$. Then, $F(\pi)$ is a facet of $P^{\leq k}_{s,t\text{-chain}}$ if and only if the minimal representative of π is

$$\begin{pmatrix} 0 & & & & & \\ & 1 & 0 & \cdots & 0 & \\ & \vdots & \ddots & \ddots & \vdots & \\ & 1 & \cdots & 1 & 0 & \\ & & & & & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & & & & & \\ & 0 & \cdots & \cdots & 0 & \\ & 1 & \ddots & & \vdots & \\ & \vdots & \ddots & \ddots & \vdots & \\ & 1 & \cdots & 1 & 0 & \\ & & & & & 0 \end{pmatrix}.$$

Lifted r-jump inequalities

r-jump inequalities (6)

$$\sum_{p=0}^{k+r-1} \sum_{q=p+1}^{k+r} \alpha_{pq} x((S_p : S_q)) \geq r,$$

where for $p < q$, $\alpha_{pq} := \min\{q - p - 1, r\}$, define facets of $\mathrm{dmt}(P^{\leq k}_{s,t\text{-chain}})$ but not of $P^{\leq k}_{s,t\text{-path}}$, $P^{\leq k}_{s,t\text{-walk}}$, or $P^{\leq k}_{s,t\text{-chain}}$. Here,

$$N = \bigcup_{p=0}^{k+r} S_p$$

is a partition of N with $S_0 = \{s\}$, and $S_{k+r} = \{t\}$.

Lifted r-jump inequalities

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where for $p < q$, $\alpha_{pq} := \min\{q-p-1, r\}$, define facets of $\text{dmt}(P^{\leq k}_{s,t\text{-chain}})$ but not of $P^{\leq k}_{s,t\text{-path}}$, $P^{\leq k}_{s,t\text{-walk}}$, or $P^{\leq k}_{s,t\text{-chain}}$. Here,

$$N = \bigcup_{p=0}^{k+r} S_p$$

is a partition of N with $S_0 = \{s\}$, and $S_{k+r} = \{t\}$.
Dahl and Gouveia (2004) investigate the problem to strengthen inequalities (6) into facet defining inequalities for $P^{\leq k}_{s,t\text{-path}}$ by decreasing coefficients of the left hand side for the cases $r \in \{1, 2\}$.

Lifted r-jump inequalities

We propose the following systematic way:

1. Derive π such that $F(\pi, +)$ is induced by r-jump inequality.
2. Derive $\sigma^{\pi}/F(\pi)$.

Lifted r-jump inequalities

- $F(\pi, +)$ facet of $\text{dmt}(P^{\le k}_{s,t\text{-chain}})$.
- $F(\pi)$ facet of $P^{\le k}_{s,t\text{-chain}}$.
- $F(\pi) \cap P^{\le k}_{s,t\text{-path}}$ facet of $P^{\le k}_{s,t\text{-path}}$ if each collection of identical rows of π has size at least two.

$$\pi^{mr} = {}^{0}\!\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 \\ \vdots & & & 2 \\ \vdots & & & \vdots \\ r-1 & & & \vdots \\ r & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r-1 \\ r & \cdots & \cdots & r \end{pmatrix}_{r}$$

Shifting vectors

Example: $k = 6$, $r = 7$:

$$\pi = {}^{0}\!\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 & 1 \\ 6 & 5 & 4 & 3 & 2 \\ 7 & 6 & 5 & 4 & 3 \\ 7 & 7 & 6 & 5 & 4 \\ 7 & 7 & 7 & 6 & 5 \\ 7 & 7 & 7 & 7 & 6 \\ 7 & 7 & 7 & 7 & 7 \end{pmatrix}_{7}, \quad \tilde{\pi} = {}^{0}\!\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \mathbf{2} & 1 & 0 & 0 & 0 \\ \mathbf{2} & \mathbf{2} & 1 & 0 & 0 \\ \mathbf{2} & \mathbf{2} & \mathbf{2} & 1 & 0 \\ 3 & \mathbf{2} & \mathbf{2} & \mathbf{2} & 1 \\ 4 & 3 & \mathbf{2} & \mathbf{2} & \mathbf{2} \\ 5 & 4 & 3 & \mathbf{2} & \mathbf{2} \\ \mathbf{6} & 5 & 4 & 3 & \mathbf{2} \\ \mathbf{6} & \mathbf{6} & 5 & 4 & 3 \\ 7 & \mathbf{6} & \mathbf{6} & 5 & 4 \\ 7 & 7 & \mathbf{6} & \mathbf{6} & 5 \\ 7 & 7 & 7 & \mathbf{6} & \mathbf{6} \\ 7 & 7 & 7 & 7 & \mathbf{6} \\ 7 & 7 & 7 & 7 & 7 \end{pmatrix}_{7}$$

Shifting vectors

$$\begin{pmatrix} 0 \\ * \\ * \\ * \\ 0 \\ 1 \\ * \\ * \\ 1 \\ 2 \\ * \\ * \\ * \\ * \\ r-1 \\ r \\ * \\ * \\ * \\ * \\ r \end{pmatrix}^{0}_{r}$$

$$v \in \mathbb{R}^n, v_1 = 0, v_{i+1} \leq v_i + 1, v_i = r, i = n-k+2, \ldots, n.$$

Shifting vectors

$$\begin{pmatrix} {}^{0} & & & \\ & 0 & 0 & \\ & * & 0 & \\ & * & * & \\ & * & * & \\ & 0 & * & \\ & 1 & 0 & \\ & * & 1 & \\ & * & * & \\ & 1 & * & \\ & 2 & 1 & \\ & * & 2 & \\ & * & * & \\ & * & * & \\ & * & * & \\ & r-1 & * & \\ & r & r-1 & \\ & * & r & \\ & * & * & \\ & * & * & \\ & * & * & \\ & r & r & \\ & & & {}_{r} \end{pmatrix}$$

$$v \in \mathbb{R}^n, v_1 = 0, v_{i+1} \leq v_i + 1, v_i = r, i = n-k+2, \ldots, n.$$

Shifting vectors

$$
\begin{pmatrix}
0 & & & & \\
 & 0 & 0 & 0 & \\
 & * & 0 & * & \\
 & * & * & 0 & \\
 & * & * & * & \\
 & 0 & * & * & \\
 & 1 & 0 & * & \\
 & * & 1 & 0 & \\
 & * & * & 1 & \\
 & 1 & * & * & \\
 & 2 & 1 & * & \\
 & * & 2 & 1 & \\
 & * & * & 2 & \\
 & * & * & * & \\
 & * & * & * & \\
 & r-1 & * & * & \\
 & r & r-1 & * & \\
 & * & r & r-1 & \\
 & * & * & r & \\
 & * & * & * & \\
 & * & * & * & \\
 & r & r & r & \\
 & & & & r
\end{pmatrix}
$$

$v \in \mathbb{R}^n, v_1 = 0, v_{i+1} \leq v_i + 1, v_i = r, i = n - k + 2, \ldots, n.$

Shifting vectors

$$
{}^{0}\begin{pmatrix}
0 & 0 & 0 & * \\
* & 0 & * & * \\
* & * & 0 & * \\
* & * & * & * \\
0 & * & * & * \\
1 & 0 & * & * \\
* & 1 & 0 & * \\
* & * & 1 & * \\
1 & * & * & * \\
2 & 1 & * & * \\
* & 2 & 1 & * \\
* & * & 2 & * \\
* & * & * & * \\
* & * & * & * \\
r-1 & * & * & * \\
r & r-1 & * & * \\
* & r & r-1 & * \\
* & * & r & * \\
* & * & * & * \\
* & * & * & * \\
r & r & r & *
\end{pmatrix}_{r}
$$

$v \in \mathbb{R}^n, v_1 = 0, v_{i+1} \leq v_i + 1, v_i = r, i = n - k + 2, \ldots, n.$

Shifting vectors

$$\begin{pmatrix} {}^{0} & & & & & \\ & 0 & 0 & 0 & * & 0 \\ & * & 0 & * & * & * \\ & * & * & 0 & * & * \\ & * & * & * & * & * \\ & 0 & * & * & * & 0 \\ & 1 & 0 & * & * & * \\ & * & 1 & 0 & * & * \\ & * & * & 1 & * & 0 \\ & 1 & * & * & * & 1 \\ & 2 & 1 & * & * & * \\ & * & 2 & 1 & * & * \\ & * & * & 2 & * & 1 \\ & * & * & * & * & 2 \\ & * & * & * & * & * \\ & r-1 & * & * & * & * \\ & r & r-1 & * & * & * \\ & * & r & r-1 & * & * \\ & * & * & r & * & r-1 \\ & * & * & * & * & r \\ & * & * & * & * & * \\ & r & r & r & * & r \\ & & & & & {}_{r} \end{pmatrix}$$

$$v \in \mathbb{R}^n, v_1 = 0, v_{i+1} \leq v_i + 1, v_i = r, i = n-k+2, \ldots, n.$$

Shifting vectors

Theorem 8

Let π be a vector whose minimal representative is a shifting vector.

(i) $F(\pi, +)$ *is a facet of* $dmt(P^{\leq k}_{s,t\text{-}chain})$.

(ii) $F(\pi)$ *is a facet of* $P^{\leq k}_{s,t\text{-}chain}$.

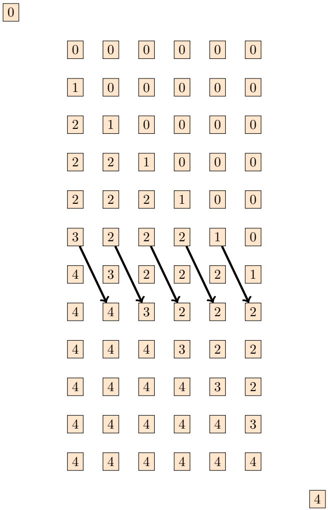
(iii) $F(\pi) \cap P^{\leq k}_{s,t\text{-}path}$ *is a facet of* $P^{\leq k}_{s,t\text{-}path}$ *if each collection of identical rows of* π *has size at least two.*

Sketch of the proof

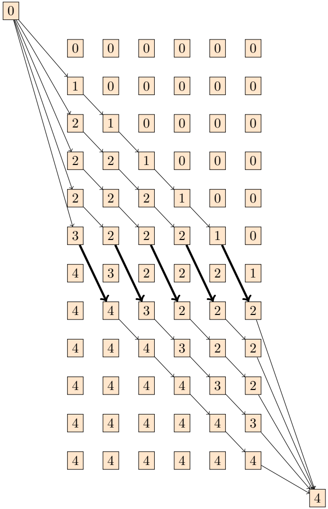
We only show that $F(\pi)$ is a high dimensional face of $P^{\leq k}_{s,t\text{-chain}}$.

- W.l.o.g. assume that π is the minimal representative of itself.
- $\dim P^{\leq k}_{s,t\text{-chain}} = |A| - (n+1) = n(n-1) + 2n - (n+1) = (n-1)(n+1) \quad \Rightarrow$ we have to show that there are $(n-1)(n+1)$ chains whose incidence vectors are in $F(\pi)$ and affinely independent.

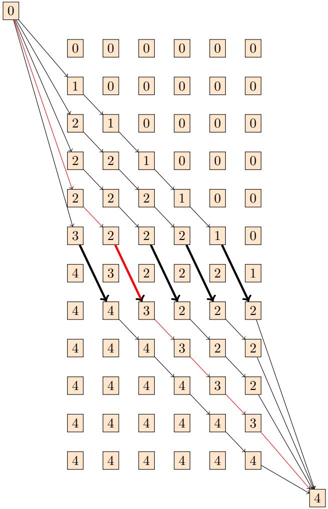
Sketch of the proof



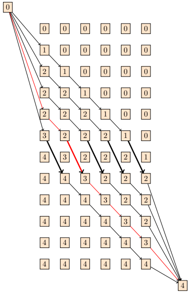
Sketch of the proof



Sketch of the proof

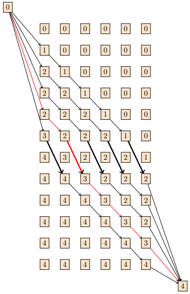


Sketch of the proof



incidence vector of $\varphi(P)$ in $F(\pi)$

Sketch of the proof



incidence vector of $\varphi(P)$ in $F(\pi)$

$\Rightarrow n(n-1)-(n-1)$ affinely independent vectors in $F(\pi)$.

Open questions

- complete linear descriptions of $P^{\leq k}_{s,t\text{-chain}}$ and its dominant
- investigation of the associated separation problems