Bounding the diameter of polytopes using oriented matroids and satisfiablity solvers

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(joint work with David Bremner, Antoine Deza, and William Hua)

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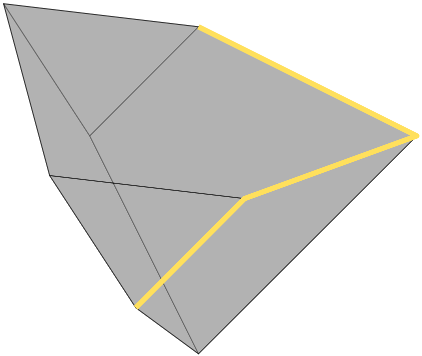

Goal

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$$\Delta(d, n) := \max\{\operatorname{diam} P : \; P \text{ is an } d\text{-polytope with } n \text{ facets}\}$$

Prove or disprove:

- $\Delta(6, 12) = 6$
- $\Delta(7, 14) = 7$

Method

- Enumerate all possible paths exceeding the desired bound
- Show that no (matroid) polytope exists with this path in the boundary

Goal

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$$\Delta(d, n) := \max\{\text{diam}\, P : \; P \text{ is an } d\text{-polytope with } n \text{ facets}\}$$

Prove or disprove:

- $\Delta(6, 12) = 6$ shown to be true
- $\Delta(7, 14) = 7$ not there yet

Method

- Enumerate all possible paths exceeding the desired bound
- Show that no (matroid) polytope exists with this path in the boundary

Diameters of Polytopes

$$\Delta(d,n) := \max\{\operatorname{diam} P : \; P \text{ is an } d\text{-polytope with } n \text{ facets}\}$$

Hirsch conjecture (1957)

$\Delta(d,n) \leqslant n - d$

d-step conjecture (Klee, Walkup, 1966)

For all $d \geqslant 2$: $\Delta(d,2d) = d$.

Lemma (Klee and Walkup, 1966)

$\Delta(d,d+k) \leqslant \Delta(k,2k)$

What is known so far?

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		$n-d$			
		4	5	6	7
d	4	4	5	5	{6,7}
	5	4	5	6	[7,9]
	6	4	5	{6,7}	[7,9]
	7	4	5	{6,7}	[7,10]

Theorem (Kalai, 1992, Kalai and Kleitman)

$\Delta(d, n) \leqslant 2(2d)^{\log_2 n}$

Theorem (Holt 2004, Fritzsche and Holt, 1999, Holt and Klee 1998)

For all $n > d \geqslant 7$: $\Delta(d, n) \geqslant n - d$

Survey

see: Kim and Santos, An update on the Hirsch conjecture, `arXiv:0907.1186`

Simplifications and reformulations

Facts

- Already simple polytopes will attain the bound
- If $n \geqslant 2d$, may assume diameter attaining path lies in more than one facet

Polar formulation

- Consider simplicial polytopes of dimension d with n vertices
- For $n \geqslant 2d$, we may assume that start and end facet of facet path have disjoint vertex sets

Enumeration of Paths

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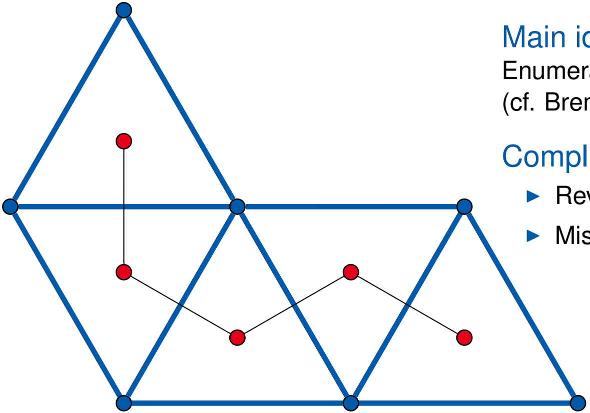

Main idea:

Enumerate possible pivot sequences (cf. Bremner, Holt, Klee (2005))

Complications

- Revisits
- Missed facets

Path Enumeration: Example with no revisits

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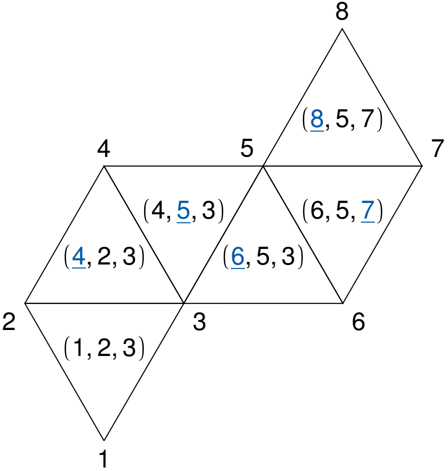

Pivot sequence

(1,4)
(2,5)
(4,6)
(3,7)
(6,8)

Canonical representation

$\langle 1, 2, 1, 3, 1 \rangle$

Oriented matroids

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Definition

Let $p_1, \cdots, p_n \in \mathbb{R}^d$ and $\widehat{p_i} = (1, p_i) \in \mathbb{R}^{d+1}$. Then we call

$$\chi(i_1, \cdots, i_{d+1}) := \operatorname{sgn}\det(\widehat{p_{i_1}}, \cdots, \widehat{p_{i_{d+1}}})$$

the *chirotope* of the point set. A chirotope is called *uniform*, if $\chi(i_1, \cdots, i_{d+1}) \neq 0$, where the i_j are pairwise disjoint.

Oriented matroids

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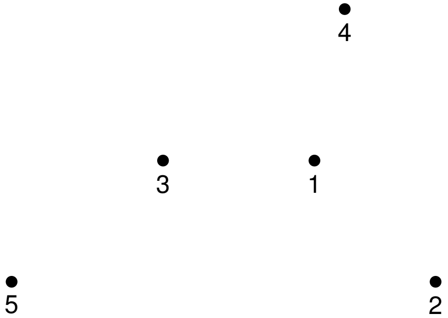
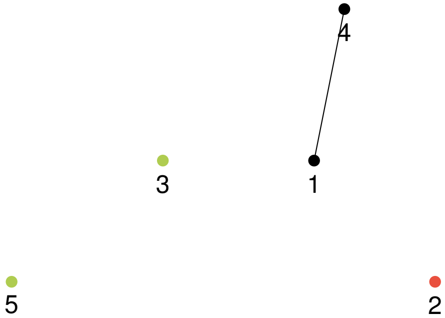


Example: $d = 2$

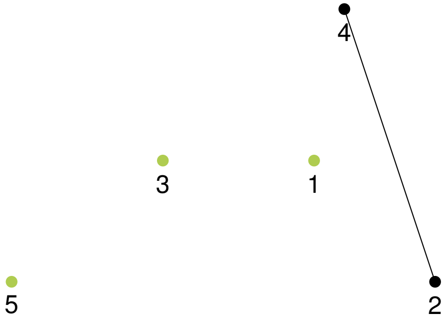
- p_i, p_j, p_k collinear
  $\Leftrightarrow \operatorname{sgn}\det(\widehat{p_i}, \widehat{p_j}, \widehat{p_k}) = 0$
- p_i, p_j, p_k ccw
  $\Leftrightarrow \operatorname{sgn}\det(\widehat{p_i}, \widehat{p_j}, \widehat{p_k}) = +$

Oriented matroids



not part of the convex hull

Oriented matroids



part of the convex hull

Oriented matroids

For a combinatorial model we use a combinatorial version of the Grassmann-Plücker relations :

$$\sum_{j=1}^{r+1} (-1)^j \det(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{r+1}) \det(x_j, y_1, \cdots, y_{r-1}) = 0$$

Oriented matroids

For a combinatorial model we use a combinatorial version of the Grassmann-Plücker relations ($r = 2$):

$$
\begin{aligned}
&\det(x_2, x_3)\det(x_1, y_1) \\
-&\det(x_1, x_3)\det(x_2, y_1) \\
+&\det(x_1, x_2)\det(x_3, y_1) = 0
\end{aligned}
$$

Oriented matroids

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For a combinatorial model we use a combinatorial version of the Grassmann-Plücker relations ($r = 2$):

$$\begin{aligned} &\chi(b,c)\,\chi(a,d)\\ -&\chi(a,c)\,\chi(b,d)\\ +&\chi(a,b)\,\chi(c,d) \end{aligned}$$

Oriented matroids

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For a combinatorial model we use a combinatorial version of the Grassmann-Plücker relations ($r = 2$):

$$\begin{aligned} \chi(b,c)\,\chi(a,d) &= 0 \\ -\chi(a,c)\,\chi(b,d) &= 0 \\ +\chi(a,b)\,\chi(c,d) &= 0 \end{aligned}$$

Oriented matroids

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For a combinatorial model we use a combinatorial version of the Grassmann-Plücker relations ($r = 2$):

$$\begin{aligned} \chi(b, c)\,\chi(a, d) &= + \\ -\chi(a, c)\,\chi(b, d) &= - \\ +\chi(a, b)\,\chi(c, d) & \end{aligned}$$

Oriented matroids

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For a combinatorial model we use a combinatorial version of the Grassmann-Plücker relations ($r = 2$):

$$\begin{aligned} &\chi(b,c)\,\chi(a,d) \\ -&\chi(a,c)\,\chi(b,d) = + \\ +&\chi(a,b)\,\chi(c,d) = - \end{aligned}$$

Oriented matroids

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The higher-dimensional case can be reduced to the linear situation via contraction

Generation of Oriented Matroids using satisfiability solvers

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Problem

Find all oriented matroids with certain additional properties

Idea

Let a satisfiability solver do the generation.

Applications

Used to show (S., 2007):

- non-existence of realizations of triangulated surfaces
- non-existence of point-line configurations

Model

Constraints

- combinatorial Grassmann-Plücker conditions
- convexity constraints
- path constraints (i.e., excluding short cuts)

Obstacles

- too many path constraints – generate them on-the-fly
- some instances take still too long – split these instances a priori

Computations

To show:	# paths	# "hard" paths
$\Delta(4,11) \neq 7$	700	0
$\Delta(4,12) \neq 8$	15500	2
$\Delta(5,12) \neq 8$	7167	21
$\Delta(6,12) \neq 7$	11	0
$\Delta(6,13) \neq 8$	745	201

Results so far

		$n-d$			
		4	5	6	7
d	4	4	5	5	**6**
	5	4	5	6	**{7,8}**
	6	4	5	**6**	[7,9]
	7	4	5	**6**	[7,10]

- $\Delta(6,12) = 6$ (Bremner and S., 2008)
- $\Delta(4,11) = 6$ (Bremner and S., 2008)
- $\Delta(4,12) = 7$ (Bremner, Deza, Hua, and S., 2009)

Still running

$\Delta(5,12)$ and $\Delta(6,13)$

Thank you!

Preprints

David Bremner, Lars Schewe
Edge-Graph Diameter Bounds for Convex Polytopes with Few Facets
`arXiv:0809.0915`

David Bremner, Antoine Deza, William Hua, and Lars Schewe
More bounds on the diameter of convex polytopes
`arXiv:0911.4982`