Symmetry Matters for the Sizes of Extended Formulations

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Aussois, January 2010

Joint work with:
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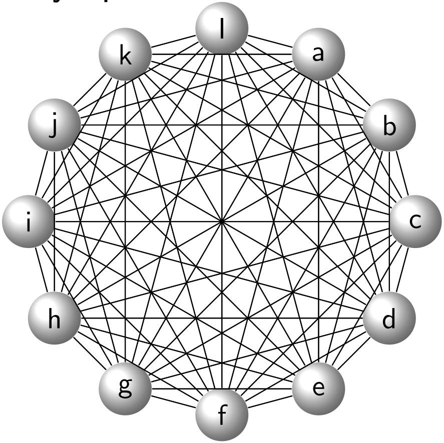
Outline

Contents

Spanning Tree Polytopes



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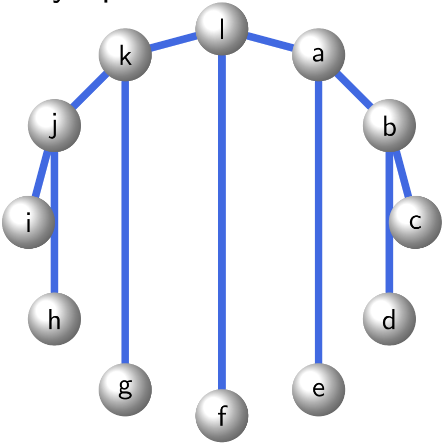

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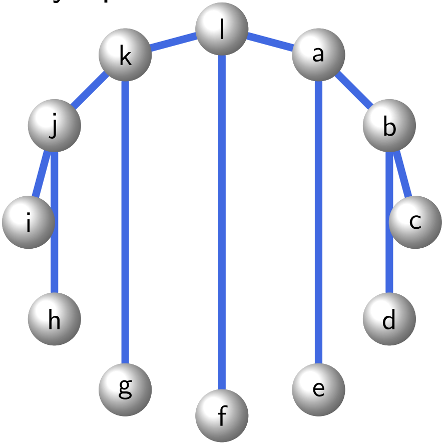

Spanning Tree Polytope on Complete Graph $K_n = (V, E)$

$$\mathrm{P}_{\mathrm{spt}}(n) = \mathrm{conv}\{\chi(T) \in \{0,1\}^E \;:\; T \subseteq E \text{ spanning tree of } K_n\}$$

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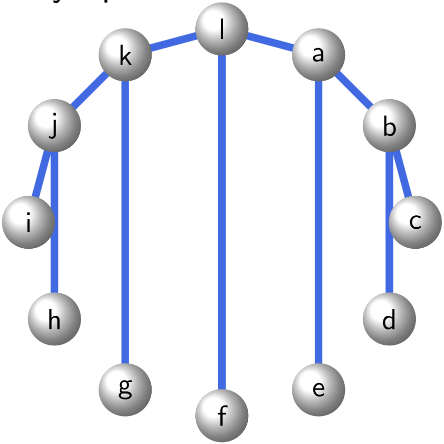

Edmonds 1971

$P_{\mathrm{spt}}(n)$ is described by $x \geq \mathbf{0}$, $x(E) = n - 1$ and

$$x(E(S)) \leq |S| - 1 \quad \text{for all } \varnothing \neq S \subseteq V.$$

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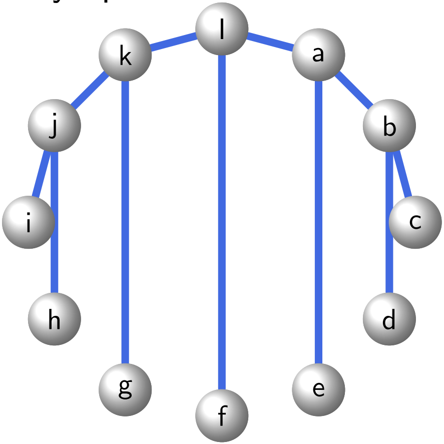

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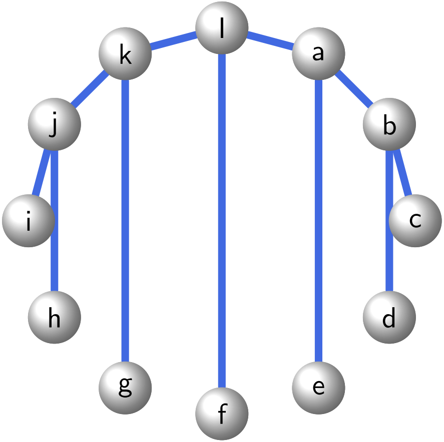
Size (#constraints + #variables): $\Theta(2^n)$

Additional Variables



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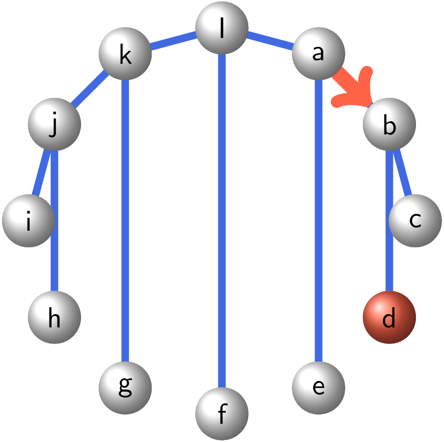

Meaning of z-variables

$z_{(a,b),d} = 1$ if $\{a, b\}$ in tree and d on side of b

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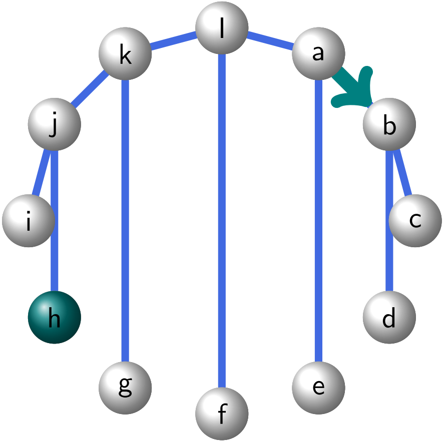

Meaning of z-Variables

$z_{(a,b),h} = 0$ if $\{a, b\}$ in tree and h not on side of b

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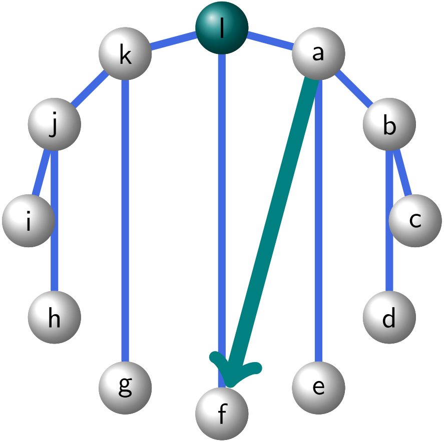

Meaning of z-Variables

$z_{(a,f),l} = 0$ if $\{a, f\}$ not in tree

Extended Formulation for $\mathrm{P}_{\mathrm{spt}}(n)$

Kipp Martin 87, Yannakakis 91, Conforti et al. 09

For $Q_{\mathrm{spt}}(n) \subseteq \mathbb{R}^d$ defined by $x \geq \mathbf{0}$, $z \geq \mathbf{0}$, $x(E) = n - 1$, and

$$x_{\{v,w\}} - z_{\{v,w\},v,u} - z_{\{v,w\},w,u} = 0 \qquad \text{for all } v, w, u \in V$$

$$x_{\{v,w\}} + \sum_{u \in [n] \setminus \{v,w\}} z_{\{v,u\},u,w} = 1 \qquad \text{for all } v, w \in V$$

we have $p(Q_{\mathrm{spt}}(n)) = \mathrm{P}_{\mathrm{spt}}(n)$, where $p : \mathbb{R}^d \to \mathbb{R}^E$ is the orthogonal projection onto the x-variables.

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Size: $\mathrm{O}\left(n^3\right)$

Extensions in General

Definition

A polyhedron $Q \subseteq \mathbb{R}^d$ and a linear projection $p : \mathbb{R}^d \to \mathbb{R}^n$

- form an **extension** of a polytope $P \subseteq \mathbb{R}^m$ if $P = p(Q)$ holds.

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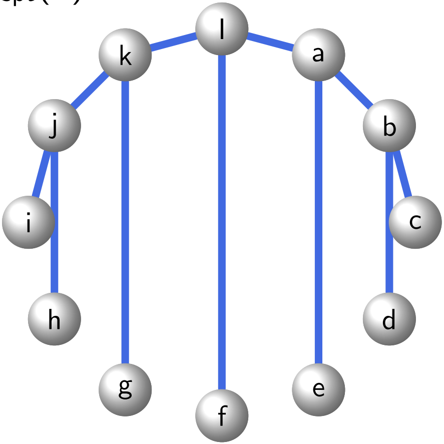
Crucial Fact

If $p(y) = Ty$ then, for each $c \in \mathbb{R}^m$, we have

$$\max\{\langle c, x\rangle \,:\, x \in P\} = \max\{\langle T^t c, y\rangle \,:\, y \in Q\}.$$

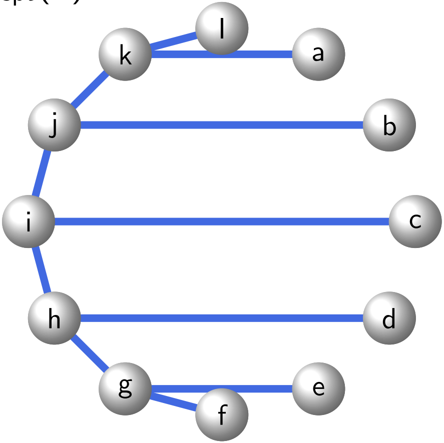
Symmetry of $\mathrm{P}_{\mathrm{spt}}(n)$



The Symmetric Group $\mathfrak{S}(n)$...

...acts on $\mathbb{R}^E$ with $\pi.\mathrm{P}_{\mathrm{spt}}(n) = \mathrm{P}_{\mathrm{spt}}(n)$ for all $\pi \in \mathfrak{S}(n)$.

Symmetry of $P_{spt}(n)$



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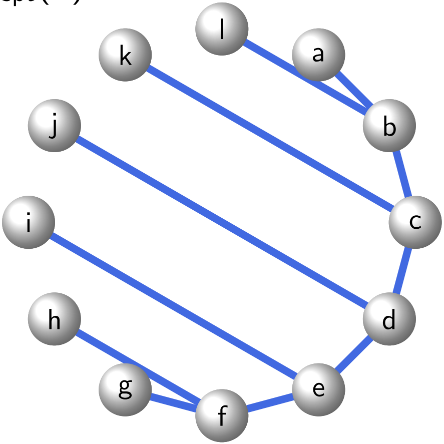

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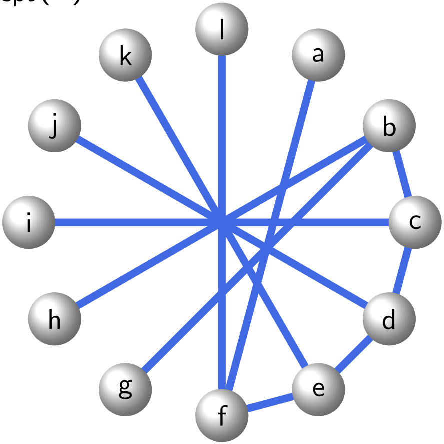

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Symmetry of the Extension of $\mathrm{P}_{\mathrm{spt}}(n)$

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An extension $Q \subseteq \mathbb{R}^d$ of $P \subseteq \mathbb{R}^m$ with projection $p : \mathbb{R}^d \to \mathbb{R}^m$ is **symmetric** (w.r.t. to G acting on $\mathbb{R}^m$ with $\pi.P = P$ for all $\pi \in G$) if, for every $\pi \in G$, there is some $\kappa_\pi \in \mathfrak{S}(d)$ with $\kappa_\pi.Q = Q$ and

$$\pi.p(y) = p(\kappa_\pi.y) \quad \text{for all } y \in \mathbb{R}^d\,.$$

Symmetry of the Extension of $\mathrm{P}_{\mathrm{spt}}(n)$

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Example

$Q_{\mathrm{spt}}(n)$ is symmetric:

$$\begin{aligned} x_{\{v,w\}} - z_{\{v,w\},v,u} - z_{\{v,w\},w,u} &= 0 \qquad \text{for all } v, w, u \in V \\ x_{\{v,w\}} + \sum_{u \in [n] \setminus \{v,w\}} z_{\{v,u\},u,w} &= 1 \qquad \text{for all } v, w \in V \end{aligned}$$

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For even n, $\mathsf{P}^{n/2}_{\mathrm{match}}(n)$ is described by $x \geq \mathbf{0}$ and:

$$x(\delta(v)) = 1 \qquad \text{for all } v \in V$$

$$x(E(S)) \leq \frac{|S|-1}{2} \qquad \text{for all } S \subseteq V, 3 \leq |S| \text{ odd}$$

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- $\ell = n$: NP-hard (Traveling Salesman Problem)

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- $\ell = n$: NP-hard (Traveling Salesman Problem)
- $\ell = \mathrm{O}(\log(n))$: polynomial (Alon, Yuster, Zwick 95)

Yannakakis 1991

Perfect Matching Polytopes

The size of every extension for $P^{n/2}_{\text{match}}(n)$ (n even) that is symmetric is bounded from below by $\Omega\binom{n}{\lfloor n/4 \rfloor} = 2^{\Omega(n)}$.

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Conjecture

"We do not think that asymmetry helps much."

Our Results (Matching Polytopes)

Lower Bounds for Symmetric Extensions

The size of every extension for $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$ $(1 \leq \ell \leq \frac{n}{2})$ that is symmetric is bounded from below by $\mathrm{const} \cdot \binom{n}{\lfloor (\ell-1)/2 \rfloor}$.

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For all n and ℓ, there are extensions for $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$ whose sizes can be bounded by $2^{\mathrm{O}(\ell)} n^2 \log(n)$.

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. . . asymmetry does help much!

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The size of every extension for $P^{\ell}_{\text{cycl}}(n)$ $(42 \le \ell \le n)$ that is symmetric is bounded from below by const $\cdot \binom{\lfloor n/3 \rfloor}{\lfloor (\lfloor \ell/6 \rfloor - 1)/2 \rfloor}$.

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Lemma (Balas 75/98)

If the polytopes $P_i \subseteq \mathbb{R}^m$ (for $i \in [q]$) have extensions Q_i of size s_i, respectively, then

$$P = \operatorname{conv}(P_1 \cup \cdots \cup P_q)$$

has an extension of size $\sum_{i=1}^{q}(s_i + 2) + 1$.

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Colorful Matchings

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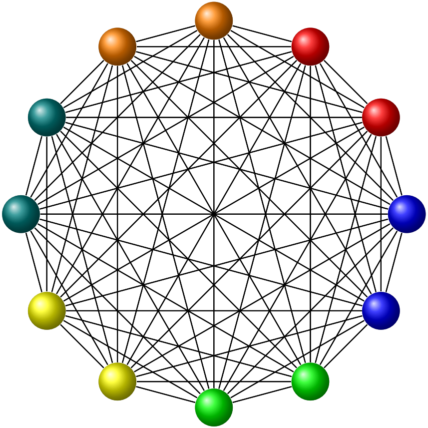
For $V = W_1 \uplus \cdots \uplus W_{2\ell}$, a matching $M \subseteq E$ is **colorful** if it matches exactly one node from each set W_i.



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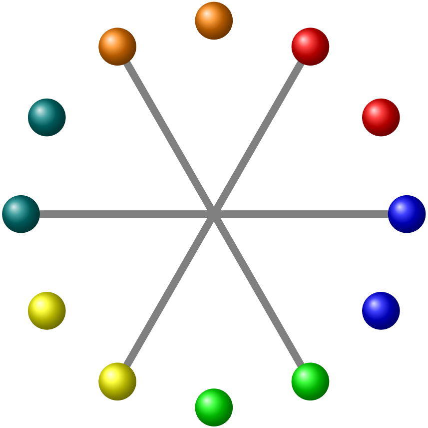

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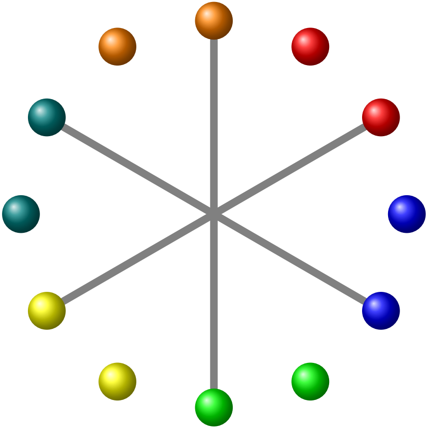

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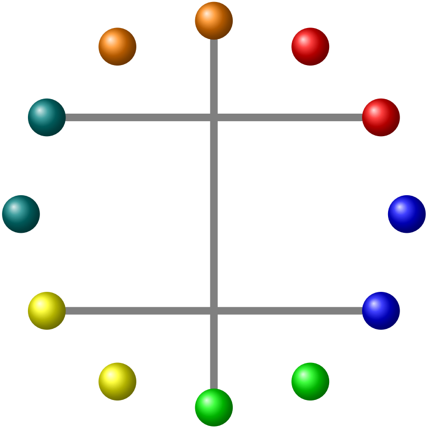


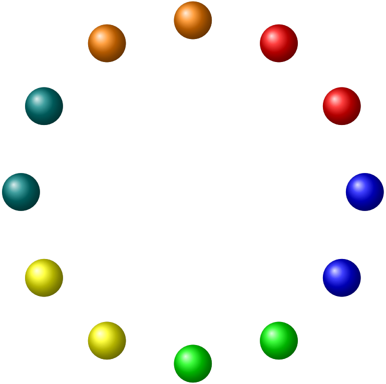
Colorful Matching Polytopes

Linear Description in $\mathbb{R}_+^E$ (special PBS [K & Loos 09])

$$x(E(W_i)) = 0 \qquad \text{for all } i \in \{1, \ldots, 2\ell\}$$

$$x(\delta(W_i)) = 1 \qquad \text{for all } i \in \{1, \ldots, 2\ell\}$$

$$x(E(\cup_{i \in S} W_i)) \leq (|S| - 1)/2 \qquad \text{for all } S \subseteq \{1, \ldots, 2\ell\}, |S| \text{ odd}$$



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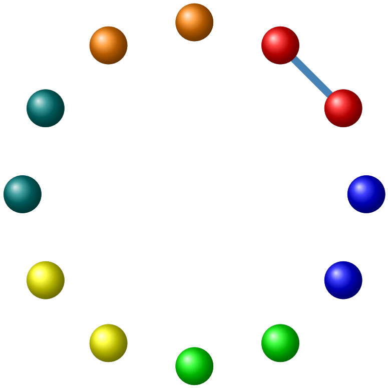

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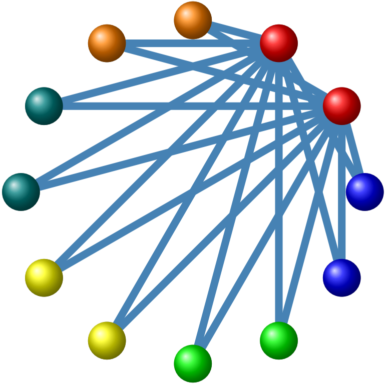

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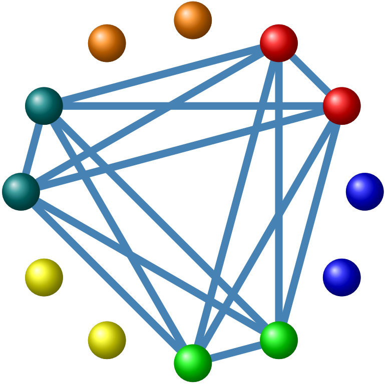
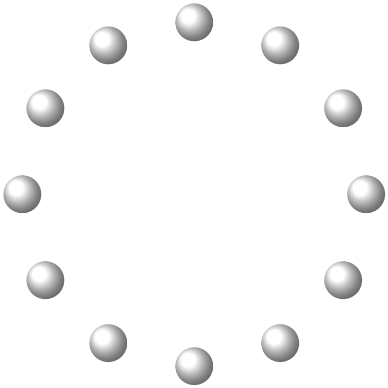


Covering $\mathcal{M}^\ell(n)$ by Colorful Matchings

(n, k)-perfect hash function family of size q

$\phi_1, \ldots, \phi_q : [n] \to [k]$ such that for all $W \subseteq [n]$ with $|W| = k$ there is some $i \in [q]$ with ϕ_i bijective on W



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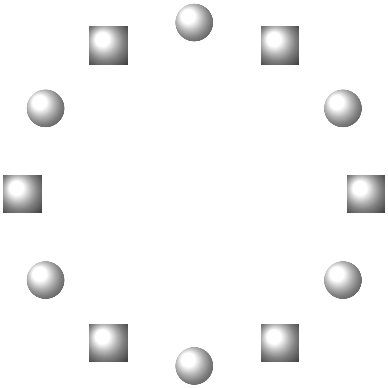

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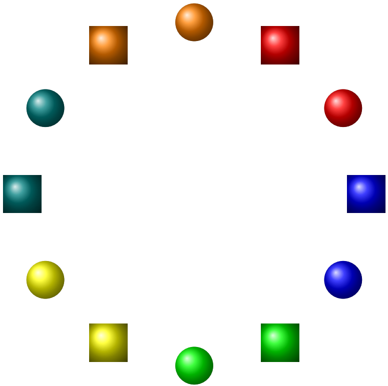

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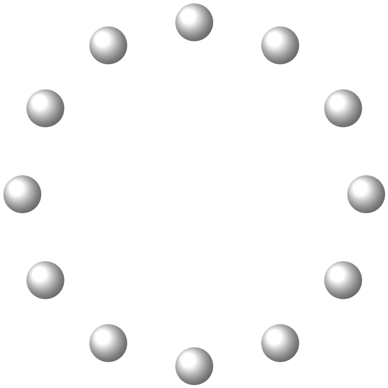

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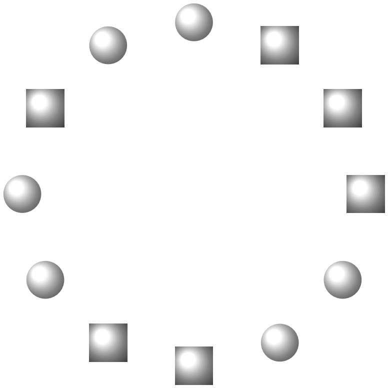

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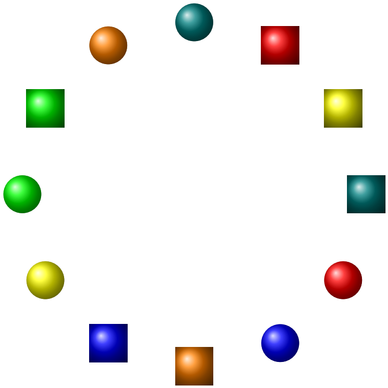

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Theorem (Alon, Yuster, Zwick 95)

For each n and k there are (n, k)-perfect hash function families of size $2^{O(k)} \log(n)$.

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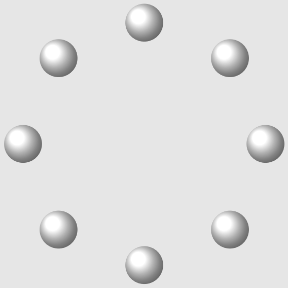

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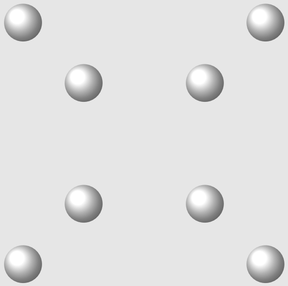

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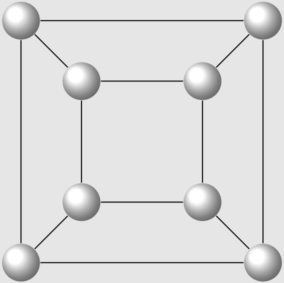

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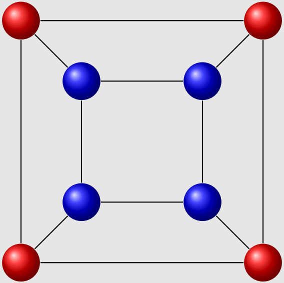

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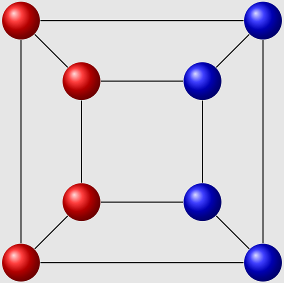

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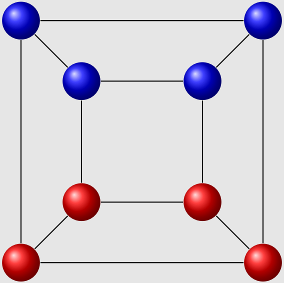

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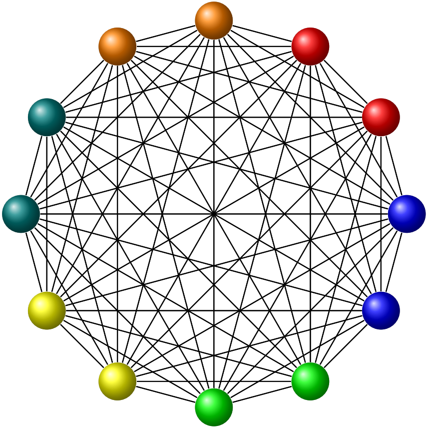

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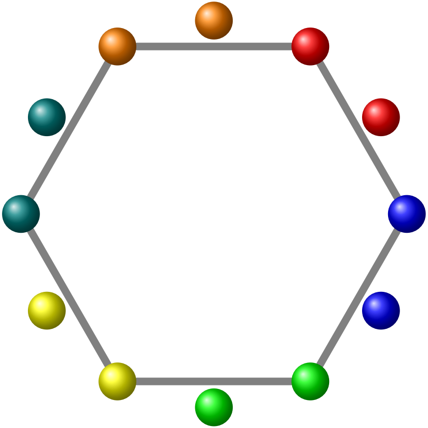

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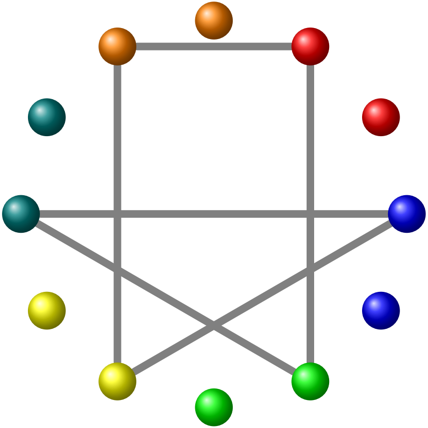
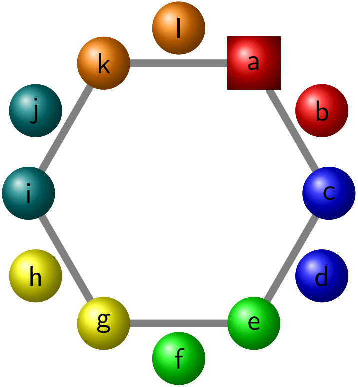


Colorful Cycles with Prescribed Node *a*



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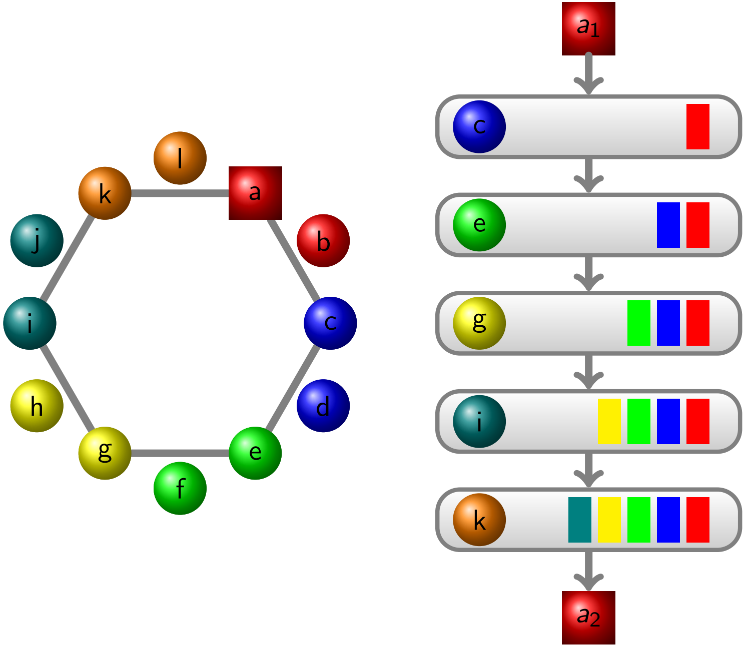

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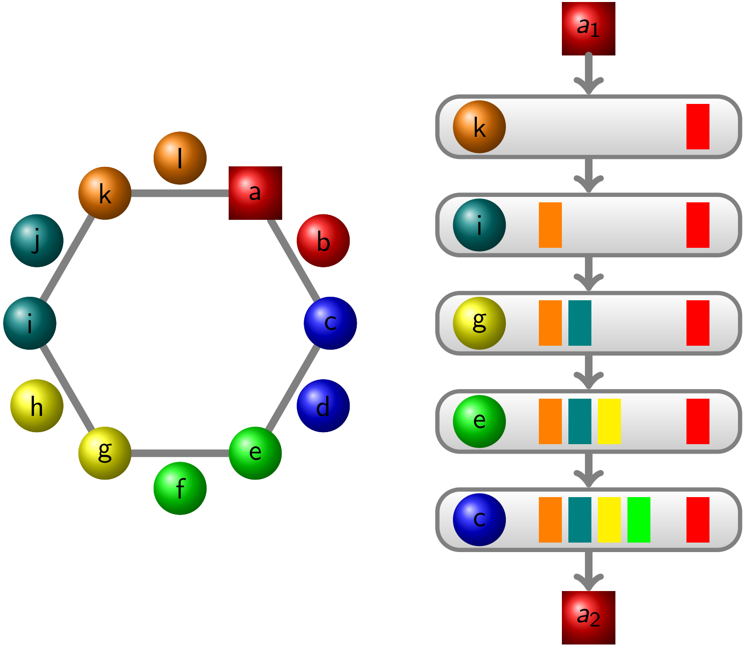

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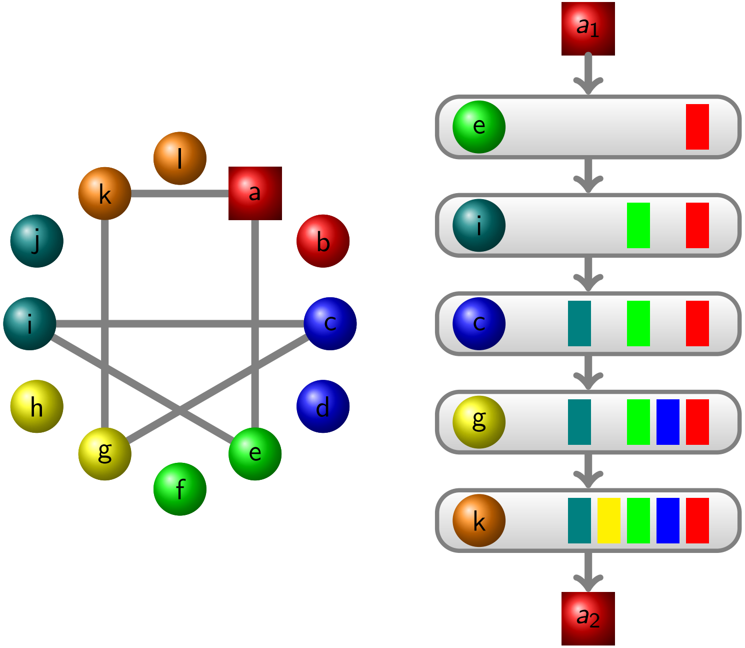

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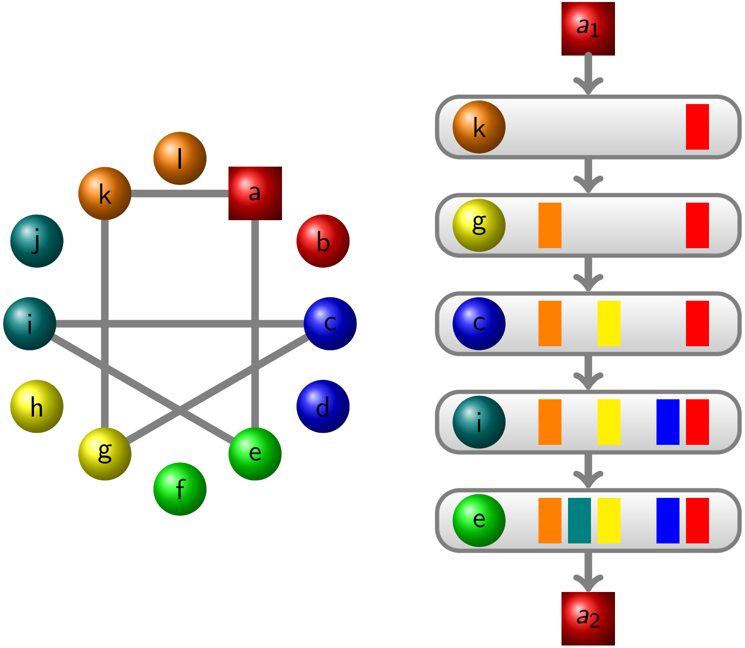

Colorful Cycle Polytopes (with Prescribed Nodes)

Extended Formulation via

- a_1-a_2 flows of value one
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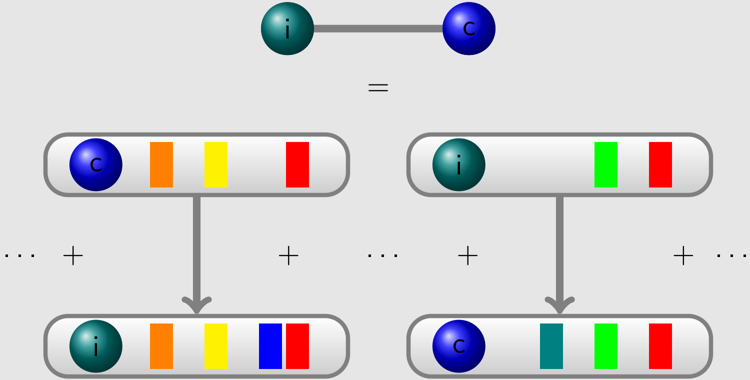

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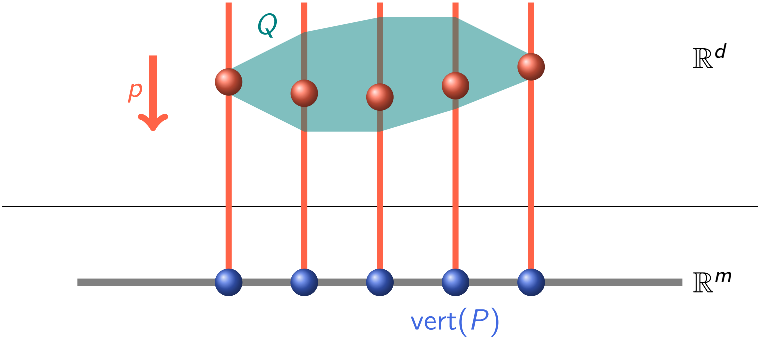
(Along the lines of Yannakakis' proof.)

Sections

Definition

A **section** for the extension $Q \subseteq \mathbb{R}^d$ of $P \subseteq \mathbb{R}^m$ with projection $p : \mathbb{R}^d \to \mathbb{R}^m$ is a map $s : \mathrm{vert}(P) \to Q$ with

$$p(s(x)) = x \quad \text{for all } x \in \mathrm{vert}(P)\,.$$

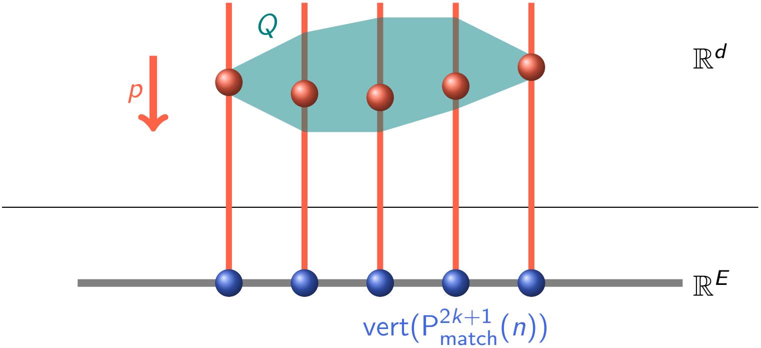


A Symmetric Section $s^\star$

Lemma

If the extension $Q \subseteq \mathbb{R}^d$ of $\mathrm{P}^{2k+1}_{\text{match}}(n)$ is symmetric, then there is a section $s^\star : \text{vert}(\mathrm{P}^{2k+1}_{\text{match}}(n)) \to \mathbb{R}^d$ such that, for each $\pi \in \mathfrak{S}(n)$, there is some $\kappa_\pi \in \mathfrak{S}(d)$ with $\kappa_\pi.Q = Q$ and

$$s^\star(\pi.x) = \kappa_\pi.s^\star(x) \quad \text{for all } x \in \text{vert}(\mathrm{P}^{2k+1}_{\text{match}}(n)), \pi \in \mathfrak{S}(n)\,.$$



Isotropy Groups (Stabilizers) of Components of $s^\star$

Lemma

The group $\mathfrak{S}(n)$ acts on $\{s_1^\star, \dots, s_d^\star\}$ such that each component function $s_j^\star : \mathrm{vert}(\mathrm{P}_{\mathrm{match}}^{2k+1}(n)) \to \mathbb{R}$ is constant on every orbit of

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Theorem (Bochert 1889)

Every primitive subgroup G of $\mathfrak{S}(n)$ with $\mathfrak{A}(n) \not\subseteq G$ satisfies

$$(\mathfrak{S}(n) : G) \ge \lfloor \tfrac{n+1}{2} \rfloor !\,.$$

Extension Q' of $P^{2k+1}_{\text{match}}(n)$ Indexed by Λ

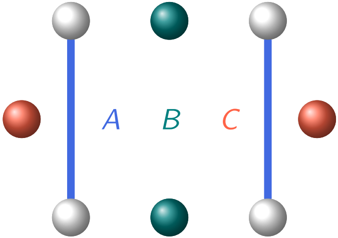
A New Extension

There is an extension $Q' = \{y \in \mathbb{R}^{d'}_{+} : A'y = b'\}$ *indexed by*

$$\Lambda = \{(A, B, C) : A \subseteq E \text{ matching}, B \cup C \subseteq V \setminus \cup A, B \cap C = \varnothing, 2|A| + |B| + |C| \leq k\}$$

with a 0/1-valued section s' with

$$s'(\chi(M))_{(A,B,C)} = 1 \quad \text{iff} \quad A \subseteq M, B \cap \cup M = \varnothing, C \subseteq \cup M.$$



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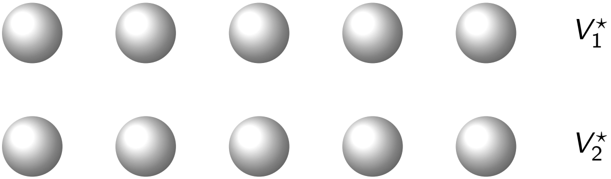

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- $\mathcal{M}^\star = \{M \in \mathcal{M}^5(n) \,:\, \cup M \subseteq V_1^\star \cup V_2^\star\}$

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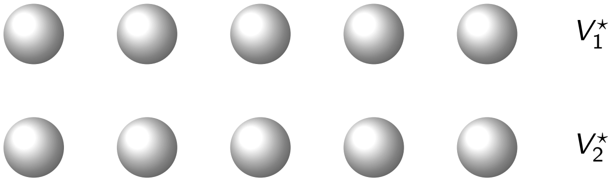

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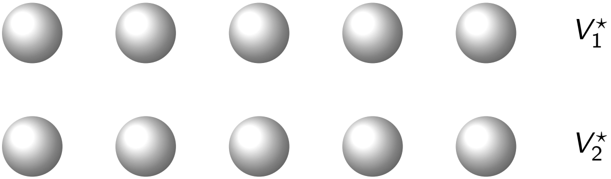

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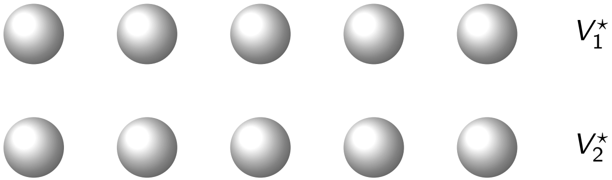

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- $x(V_1^\star : V_2^\star) \geq 1$ valid for F, contradicting $p'(y^\star)_{(V_1^\star : V_2^\star)} = \mathbf{0}$

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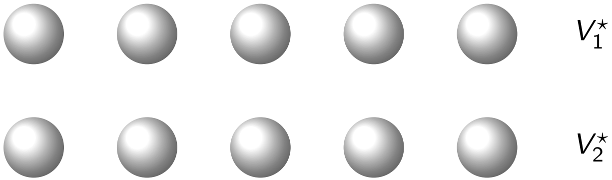

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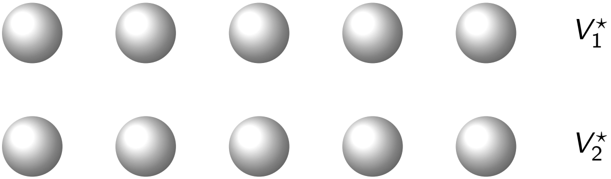

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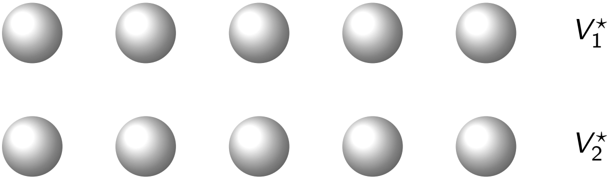

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- $\mathcal{M}_i^\star = \{M \in \mathcal{M}^\star : |M \cap (V_1^\star : V_2^\star)| = i\}$ for $i = 1, 3, 5$
- Choose coefficient α_i for all $s'(\chi(M)) \in \{0,1\}^{d'}$, $M \in \mathcal{M}_i^\star$.

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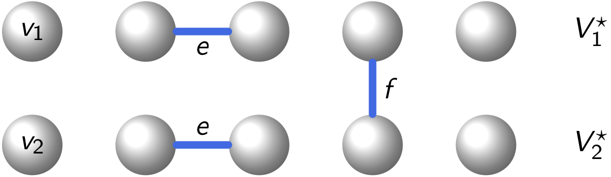

$\#\{M \in \mathcal{M}_i^\star \,:\, s'(\chi(M))_{(A,B,C)} = 1\}$

(A, B, C)	$i = 1$	$i = 3$	$i = 5$
$(\varnothing, \star, \{v_1\})$	225	600	120
$(\varnothing, \star, \{v_1, v_2\})$	216	528	96
$(\{e\}, \star, \varnothing)$	45	60	0
$(\{f\}, \star, \varnothing)$	9	72	24

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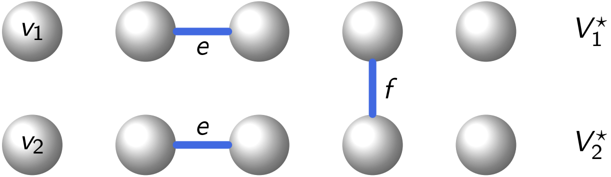

The LP for the coefficients

$$
\begin{array}{rcrcrcr}
225\alpha_1 & + & 600\alpha_3 & + & 120\alpha_5 & \geq & 0 \\
216\alpha_1 & + & 528\alpha_3 & + & 96\alpha_5 & \geq & 0 \\
45\alpha_1 & + & 60\alpha_3 & + & & \geq & 0 \\
9\alpha_1 & + & 72\alpha_3 & + & 24\alpha_5 & \geq & 0 \\
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\end{array}
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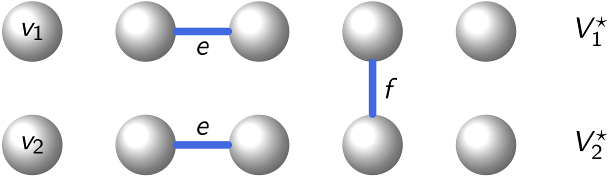

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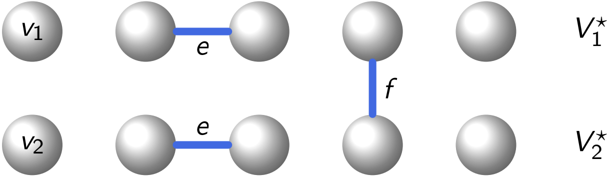

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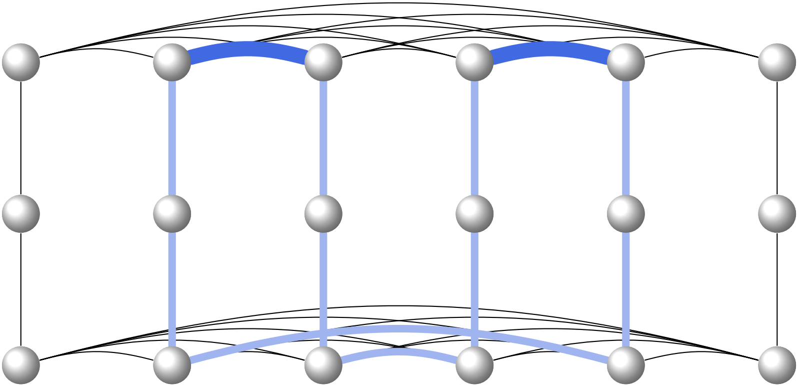
Solution: $\alpha_1 = \frac{1}{150}$, $\alpha_3 = -\frac{1}{1200}$, $\alpha_5 = 0$

Matching Polytopes as Faces of Cycle Polytopes



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Thank you for your attention.