Approximate LP Solution w/o Approximate Separation Oracle: the Case of 2-Dimensional Bin Packing

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joint work with Nikhil Bansal, Klaus Jansen, Lars Prädel and Maxim Sviridenko

Aussois 2010

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Short Summary: 2nd attempt to give my first powerpoint presentation . . .

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Further progress was "almost" made in 2005:

Theorem A [C. and Sviridenko] *If there exists a PTAS for 2-Dim Fractional Bin Packing, then there exists a polynomial-time $1.525\ldots$-approximation algorithm for 2-Dim Bin Packing*

PTAS: For any fixed $\varepsilon > 0$, finds a solution of value at most $(1+\varepsilon)\text{opt}$ in polynomial time (at least $(1-\varepsilon)\text{opt}$ for a maximization problem)

2-Dim Fractional Bin Packing

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2-Dim Bin Packing can be formulated as the following Set Covering problem:

$$\min \quad \sum_{C \in \mathcal{C}} x_C$$

$$\sum_{C \ni i} x_C \geq 1 \qquad (i \in I)$$

$$x_C \geq 0, \textbf{integer} \qquad (C \in \mathcal{C})$$

- I: set of items
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2-Dim Fractional Bin Packing: continuous relaxation of this Set Covering problem

- $O(2^{|I|})$ variables: column generation needed
- key relaxation for the practical solution of 2-Dim Bin Packing

2-Dim Fractional Bin Packing (cont.d)

The dual of 2-Dim Fractional Bin Packing is:

$$\max \quad \sum_{i \in I} \pi_i$$

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Dual Separation (or Column Generation) Problem:

- Given $\pi^* \in \mathbb{R}_+^{|I|}$, find $C^* \in \mathcal{C}$ such that $\sum_{i \in C^*} \pi_i^* > 1$

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- **Recognition Version** of 2-Dim Knapsack (in **Optimization Version:** pack a maximum-profit subset of I)

2-Dim Fractional Bin Packing (cont.d)

Connection between separation and optimization from the 1980s:

Theorem [Grötschel, Lovász and Schrijver] *There exists a polynomial-time algorithm for 2-Dim Fractional Bin Packing if there exists a polynomial-time algorithm that, given π^* such that*

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finds a solution to the dual separation problem

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Cannot be applied: 2-Dim Knapsack is strongly NP-complete (also in Recognition Version)

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Connection between approximate separation and approximate optimization from the 1990s:

Theorem [Plotkin, Shmoys and Tardos] *There exists a PTAS for 2-Dim Fractional Bin Packing if, for any $\varepsilon > 0$, there exists a polynomial-time algorithm that, given π^* such that*

$$\max_{C \in \mathcal{C}} \sum_{i \in C} \pi_i^* \geq 1 + \varepsilon,$$

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Cannot be applied at present: the existence of a PTAS for 2-Dim Knapsack is **open**!

Any Hope w/o a PTAS for 2-Dim Knapsack?

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In fact, an Asymptotic PTAS (APTAS) for 2-Dim Fractional Bin Packing would suffice

APTAS: For any fixed $\varepsilon > 0$, finds a solution of value at most $(1 + \varepsilon)\mathrm{opt} + \delta(\varepsilon)$ in polynomial time

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Moreover, for a 2-Dim Knapsack instance I let:

- p_i: profit of item i (π_i^* for dual separation)
- a_i: area of item i

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Theorem B *For any fixed $r \geq 1$, there exists a PTAS for 2-Dim Knapsack* **restricted** *to instances I for which $p_i/a_i \in [1, r]$ for $i \in I$*

In other words, if the range of the profit/area ratios is bounded by a constant we have a PTAS

(Special Case: PTAS to maximize the area packed ($p_i = a_i$ for $i \in I$), **open** before this work!)

How to use the PTAS of Theorem B?

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. . . New Idea: **Variant of dual** 2-Dim Fractional Bin Packing **with bounds on dual variables**:

$$\max \quad \sum_{i \in I} \pi_i$$

$$\sum_{i \in C} \pi_i \leq 1 \qquad (C \in \mathcal{C})$$

$$0 \leq \pi_i \leq 4a_i \quad (i \in I)$$

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Associated with **variant of primal**:

$$\min \quad \sum_{C \in \mathcal{C}} x_C + \sum_{i \in I} 4a_i y_i$$
$$\sum_{C \ni i} x_C + y_i \geq 1 \qquad (i \in I)$$
$$x_C, y_i \geq 0 \qquad (C \in \mathcal{C}, i \in I)$$

How to use the PTAS of Theorem B? (cont.d)

Lemma 1 *There exists a PTAS for the* **variant** *of* 2*-Dim Fractional Bin Packing*

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Proof. Here is a polynomial-time algorithm that, given $\pi_i^* \in [0, 4a_i]^{|I|}$ such that $\max_{C \in \mathcal{C}} \sum_{i \in C} \pi_i^* \geq 1 + \varepsilon$, finds a solution to the dual separation problem:

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- Remove all the items $i \in I$ such that $\pi_i^* \leq (\varepsilon/4)a_i$ (overall contribution to profit $\leq \varepsilon/4$)

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Lemma 2 *Given any solution of the* **variant** *of value z^*, we can obtain in polynomial time a solution of the* **original** *2-Dim Fractional Bin Packing of value at most z^*+2*

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Lemma 2 *Given any solution of the* **variant** *of value* z^**, we can obtain in polynomial time a solution of the* **original** 2*-Dim Fractional Bin Packing of value at most* z^*+2

Proof. Consider a solution (x_C^*, y_i^*) of the variant and let $I^* := \{i \in I : y_i^* > 0\}$:

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- Pack the items in I^* into bins by Next-Fit Decreasing Height (NFDH)

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- If at least one bin contains items $S^* \subseteq I^*$ such that $\sum_{i \in S^*} a_i \geq 1/4$, let $\alpha := \min_{i \in S^*} y_i^*$, and define the new solution $y_i^* := y_i^* - \alpha \ (i \in I^*)$, $x_{S^*} := x_{S^*} + \alpha$ (feasible, value $\leq z^*$)

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- **Repeat** until no bin packed by NFDH has area occupied $\geq 1/4$

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Proof. Consider a solution (x_C^*, y_i^*) of the variant and let $I^* := \{i \in I : y_i^* > 0\}$:

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- **Repeat** until no bin packed by NFDH has area occupied $\geq 1/4$

At the end, at most two bins are packed by NFDH, say with items S_1^* and S_2^*: define the final solution $y_i^* := 0 \ (i \in I^*)$, $x_{S_1^*} := x_{S_2^*} := 1$ (feasible, value $\leq z^*{+}2$) □

How to use the PTAS of Theorem B? (cont.d)

Lemmas 1 and 2 imply:

Theorem *There exists an APTAS for 2-Dim Fractional Bin Packing*

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which in turn by Theorem A' implies:

Corollary *There exists a polynomial-time $1.525\ldots$-approximation algorithm for 2-Dim Bin Packing*

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Structural Lemma leading to the PTAS in Theorem B: given an "unstructured" packing of items (left), proves the existence of a corresponding "structured" packing of the same items (right)

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